Characterization of Critical Values of Branching Random Walks on Weighted Graphs through Infinite-Type Branching Processes

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Abstract We study the branching random walk on weighted graphs; site-breeding and edge-breeding branching random walks on graphs are seen as particular cases. Two kinds of survival can be identified: a weak survival (with positive probability there is at least one particle alive somewhere at any time) and a strong survival (with positive probability the colony survives by returning infinitely often to a fixed site). The behavior of the process depends on the value of a certain parameter which controls the birth rates; the threshold between survival and (almost sure) extinction is called critical value. We describe the strong critical value in terms of a geometrical parameter of the graph. We characterize the weak critical value and relate it to another geometrical parameter. We prove that, at the strong critical value, the process dies out locally almost surely; while, at the weak critical value, global survival and global extinction are both possible.

1 Introduction

We consider the branching random walk (briefly BRW) as a continuous-time process where particles live on an at most countable set X (the set of sites). Each particle lives on a site and, independently of the others, has a random lifespan; during its life it breeds at random intervals and sends its offspring to randomly chosen sites. More precisely each particle has an exponentially distributed lifespan with mean 1. To a particle living at site x, for any $y \in X$, there corresponds a Poisson clock of rate λk_{xy}: when the clock rings, a new particle

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is born in y (where $K = (k_{xy})_{x,y \in X}$ is a matrix with nonnegative entries and $\lambda > 0$), provided that the particle at x is still alive.

This approach unifies the two main points of view which may be found in the literature: the *site-breeding* BRW and the most widely used *edge-breeding* BRW. Indeed in the first case there is a constant reproduction rate λ at each site and the offspring is sent accordingly to a probability distribution on X (thus $(k_{xy})_{x,y \in X}$ is a stochastic matrix). Examples can be found in [1, 4, 11] (in the later it is called *modified* BRW). In the edge-breeding model, X is a graph and to each (oriented) edge one associates a reproduction rate λ (thus $(k_{xy})_{x,y \in X}$ is the adjacency matrix of the graph). Some examples are in [1, 6, 9–11]. On regular graphs (see for instance [5, 7]), the site-breeding model employing the transition matrix of the simple random walk is equivalent, up to a multiplicative constant, to the edge-breeding one.

We consider the BRW with initial configuration given by a single particle at a fixed site x: there are two kinds of survival:

 (i) *weak* (or global) *survival*—the total number of particles is positive at each time;
(ii) *strong* (or local) *survival*—the number of particles at site x is not eventually 0.

Let us denote by $\lambda_w(x)$ (resp. $\lambda_s(x)$) the infimum of the values of λ such that there is weak (resp. strong) survival with positive probability. Clearly $\lambda_w(x) \leq \lambda_s(x)$ and these values do not depend on x when K is irreducible (see Sect. 2.1). For the edge-breeding BRW on a connected graph, in [10] it was proved that $\lambda_s = 1/M_s$ where M_s is a geometrical parameter of the graph. This result can be extended to the BRW on weighted graphs (Theorem 4.1). To our knowledge the behavior of the BRW at $\lambda = \lambda_s(x)$ was yet unknown: we prove that there is almost sure extinction in Theorem 4.7 (we proved the same result for BRW on multigraphs in [1]).

More challenging is the characterization of the weak critical parameter $\lambda_w(x)$ and the study of the weak critical behavior. Following the ideas which lead to the characterization of $\lambda_s(x)$ one naturally guesses that $\lambda_w(x) = 1/M_w(x)$ (see Sect. 2.1 for the definition). Indeed in [1] we proved that, when K is irreducible, $\lambda_w \geq 1/M_w$ and we gave sufficient conditions for equality (for instance all site-breeding BRWs satisfy these conditions). In this paper we use a different approach which allows us to characterize $\lambda_w(x)$ in terms of the existence of solutions of certain infinite-dimensional linear systems (Theorem 4.2); in particular we show that $\lambda_w(x)$ is related to the so-called Collatz-Wielandt numbers of some linear operator (see [2, 3, 8] for the definition).

Thanks to this characterization, we prove a stronger lower bound, $\lambda_w(x) \geq 1/M_w^-(x)$ and give sufficient conditions for equality (Remark 4.4 and Propositions 4.5 and 4.6). We show (Example 2) that it may be that $\lambda_w(x) = 1/M_w^-(x) \neq 1/M_w(x)$. As for the critical behavior, Example 3 is a BRW which globally survives at $\lambda_w(x)$ (while for instance on finite weighted graphs the BRW dies out at $\lambda_w(x)$—this is a particular case of Theorem 4.8). The question whether $\lambda_w(x) = 1/M_w^-(x)$ always holds is, as far as we know, still open.

The basic idea behind the study of $\lambda_w(x)$ relies on the comparison between the BRW and an infinite-type branching process (briefly IBP). It is well known that the probability of extinction of a Galton-Watson branching process is the smallest positive fixed point of a certain generating function. In Sect. 3 we prove some results on IBPs by studying an infinite-dimensional generating function and its fixed points.

The paper is organized as follows: Sects. 2.1 and 2.2 introduce the basic definitions (among which the definition of weighted graph and of the geometrical parameters of the graph). In Sect. 2.3 we prove some results on fixed points for monotone functions in partially ordered sets. In Sect. 3 we define IBPs and associate in a "canonical" way an IBP

to a given BRW. Section 4 is devoted to the study of the critical values $\lambda_s(x)$ and $\lambda_w(x)$ (Sect. 4.1) and of the strong and weak critical behaviors (Sect. 4.2). Finally in Sect. 5 we give some examples of IBPs and BRWs.

2 Basic Definitions and Preliminaries

2.1 Weighted Graphs

Let us consider (X, K) where X is a countable (or finite) set and $K = (k_{xy})_{x,y \in X}$ is a matrix of nonnegative *weights* (that is, $k_{xy} \geq 0$) such that $\sup_{x \in X} \sum_{y \in X} k_{xy} = M < \infty$. We denote by (X, K) the *weighted graph* with set of edges $E(X) := \{(x, y) \in X \times X : k_{xy} > 0\}$, where to each edge (x, y) we associate the weight k_{xy}. We say that K is *irreducible* if $(X, E(X))$, as a directed graph, is connected.

We define recursively $k_{xy}^n := \sum_{w \in X} k_{xw}^{n-1} k_{wy}$ (where $k_{xy}^0 := \delta_{xy}$); moreover we set $T_x^n := \sum_{y \in X} k_{xy}^n$ and $\phi_{xy}^n := \sum_{x_1,\ldots,x_{n-1} \in X \setminus \{y\}} k_{xx_1} k_{x_1 x_2} \cdots k_{x_{n-1} y}$; by definition $\phi_{xy}^0 := 0$ for all $x, y \in X$. Clearly k_{xy}^n is the total weight of all paths of length n from x to y, T_x^n is the total weight of all paths of length n from x, while ϕ_{xy}^n is the analog of k_{xy}^n regarding only paths reaching y for the first time at the n-th step.

For k_{xy}^n and T_x^n the following recursive relations hold for all $n, m \geq 0$

$$k_{xy}^{n+m} = \sum_{w \in X} k_{xw}^n k_{wy}^m; \qquad \begin{cases} T_x^{n+m} = \sum_{w \in X} k_{xw}^n T_w^m, \\ T_x^0 = 1 \end{cases}$$

and, for all $n \geq 1$,

$$k_{xy}^n = \sum_{i=0}^n \phi_{xy}^i k_{yy}^{n-i}.$$

Whenever, given $x, y \in X$, there exists $n \in \mathbb{N}$ such that $k_{xy}^n > 0$ we write $x \to y$; if $x \to y$ and $y \to x$ then we write $x \leftrightarrow y$. This is an equivalence relation; let us denote by $[x]$ the equivalence class of x (usually called *irreducible class*). We observe that the summations involved in k_{xx}^n could be equivalently restricted to sites in $[x]$, moreover $\lambda_s(x)$ depends only on $[x]$. Similarly one can prove that $\lambda_w(x)$ depends only on $[x]$.

We introduce the following geometrical parameters

$$M_s(x, y; X) := \limsup_n (k_{xy}^n)^{1/n}, \qquad M_w(x; X) := \limsup_n (T_x^n)^{1/n},$$

$$M_w^-(x; X) := \liminf_n (T_x^n)^{1/n}.$$

In the rest of the paper, whenever there is no ambiguity, we will omit the dependence on X. Moreover, we write $M_s(x) := M_s(x, x)$; supermultiplicative arguments imply that $M_s(x) = \lim_n (k_{xx}^{dn})^{1/dn}$ for some $d \in \mathbb{N}$ hence, for all $x \in X$, we have that $M_s(x) \leq M_w^-(x) \leq M_w(x)$. It is easy to show that the above quantities are constant within an irreducible class; hence in the irreducible case the dependence on x, y will be omitted.

2.2 Generating Functions

Let us consider the following generating functions

$$\Gamma(x, y|\lambda) := \sum_{n=0}^{\infty} k_{xy}^n \lambda^n, \qquad \Theta(x|\lambda) := \sum_{n=0}^{\infty} T_x^n \lambda^n, \qquad \Phi(x, y|\lambda) := \sum_{n=1}^{\infty} \phi_{xy}^n \lambda^n;$$

note that the radii of convergence of $\Gamma(x, y|\lambda)$ and $\Theta(x|\lambda)$ are $1/M_s(x, y)$ and $1/M_w(x)$ respectively. The following relation holds

$$\Gamma(x, y|\lambda) = \Phi(x, y|\lambda)\Gamma(y, y|\lambda) + \delta_{xy}, \quad \forall \lambda : |\lambda| < \min(1/M_s(x, y), 1/M_s(y)). \quad (2.1)$$

Since

$$\Gamma(x, x|\lambda) = \frac{1}{1 - \Phi(x, x|\lambda)}, \quad \forall \lambda \in \mathbb{C} : |\lambda| < 1/M_s(x), \quad (2.2)$$

we have that $1/M_s(x) = \max\{\lambda \geq 0 : \Phi(x, x|\lambda) \leq 1\}$ for all $x \in X$ (see Sect. 2.2 of [1] for details).

2.3 Fixed Points in Partially Ordered Sets

Let $(Q, \geq)$ be a partially ordered set and $W : Q \mapsto Q$ be a nondecreasing function, that is, $x \geq y$ implies $W(x) \geq W(y)$. Let us denote by $(-\infty, y]$ and $[y, +\infty)$ the *intervals* $\{w \in Q : w \leq y\}$ and $\{w \in Q : w \geq y\}$ respectively. We consider a topology τ on Q such that all the intervals $(-\infty, y]$ and $[y, +\infty)$ are closed.

Proposition 2.1 *Let $W : Q \mapsto Q$ be a nondecreasing function.*

(a) *If $q \geq W(q)$ then $W((-\infty, q]) \subseteq (-\infty, q]$. If $q \leq W(q)$ then $W([q, +\infty)) \subseteq [q, +\infty)$.*

Moreover let us suppose that $q_0 \in Q$ satisfies $W(q_0) \geq q_0$ (resp. $W(q_0) \leq q_0$) and define the sequence $\{q_n\}_{n \in \mathbb{N}}$ recursively by $q_{n+1} = W(q_n)$, for all $n \in \mathbb{N}$. The following hold.

(b) *The sequence is nondecreasing (resp. nonincreasing).*
(c) *If the sequence has a cluster point q and y is such that $y \geq q_0$, $y \geq W(y)$ (resp. $y \leq q_0$, $y \leq W(y)$) then $q \leq y$ (resp. $q \geq y$).*
(d) *Every cluster point q of $\{q_n\}_{n \in \mathbb{N}}$ satisfies $q \geq q_0$ (resp. $q \leq q_0$). If W is continuous then there is at most one cluster point q and*

$$W(q) = q \quad and \quad (-\infty, q] = \bigcap_{y \geq q_0 : W(y) \leq y} (-\infty, y] = \bigcap_{y \geq q_0 : W(y) = y} (-\infty, y]$$

$$\left(resp. \ W(q) = q \quad and \quad (-\infty, q] = \bigcup_{y \leq q_0 : W(y) \geq y} (-\infty, y] = \bigcup_{y \leq q_0 : W(y) = y} (-\infty, y] \right).$$

Proof (a), (b) and (c) are straightforward. As for (d), the first claim follows since $[q_0, +\infty)$ is closed. Continuity implies that for every cluster point $W(q) = q$. Moreover if q and $\widetilde{q}$ are two cluster points then since $q_0 \leq \widetilde{q}$ then by (c) $q \leq \widetilde{q}$ and similarly $\widetilde{q} \leq q$ whence $q = \widetilde{q}$. By (c) $(-\infty, q] = \bigcap_{y \geq q_0 : W(y) \leq y}(-\infty, y]$. Moreover since $W(q) = q$

$$(-\infty, q] \supseteq \bigcap_{y \geq q_0 : W(y) = y} (-\infty, y] \supseteq \bigcap_{y \geq q_0 : W(y) \leq y} (-\infty, y]$$

whence the claim. The proof of the second claim is analogous.   □

The following corollary is a direct consequence of the previous proposition; the proof is straightforward and we omit it.

Corollary 2.2 *Let Q have a smallest element $\mathbf{0}$ (resp. a largest element $\mathbf{1}$), $W : Q \mapsto Q$ be a continuous nondecreasing function. If $\{q_n\}_{n\in\mathbb{N}}$ is recursively defined by*

$$\begin{cases} q_{n+1} = W(q_n), \\ q_0 = \mathbf{0} \quad (\text{resp. } q_0 = \mathbf{1}). \end{cases} \tag{2.3}$$

then $\{q_n\}_{n\in\mathbb{N}}$ has at most one cluster point q; moreover q is the smallest (resp. largest) fixed point of W and for any $y \in Q$, we have that $q < y$ (resp. $q > y$) if and only if there exists $y' < y$ (resp. $y' > y$) such that $W(y') \leq y'$ (resp. $W(y') \geq y'$).

3 Infinite-Type Branching Processes

Let X be a set which is at most countable. Each element of this set represents a different type of particle of a (possibly) infinite-type branching process. Given $f \in \Psi := \{g \in \mathbb{N}^X : S(g) := \sum_{x\in X} g(x) < +\infty\}$, at the end of its life a particle of type x gives birth to $f(y)$ children of type y (for all $y \in X$) with probability $\mu_x(f)$ where $\{\mu_x\}_{x\in X}$ is a family of probability distributions on the (countable) measurable space $(\Psi, 2^\Psi)$.

To the family $\{\mu_x\}_{x\in X}$ we associate a generating function $G : [0,1]^X \to [0,1]^X$ which can be considered as an infinite dimensional power series. More precisely, for all $z \in [0,1]^X$ the function $G(z) \in [0,1]^X$ is defined as follows

$$G(z|x) := \sum_{f\in\Psi} \mu_x(f) \prod_{y\in X} z(y)^{f(y)}. \tag{3.1}$$

Note that G is continuous with respect to the *pointwise convergence topology* (or *product topology*) on $[0,1]^X$. Indeed, every $f \in \Psi$ is finitely supported, hence $\prod_{y\in X} z(y)^{f(y)}$ is a finite product and $z \mapsto \prod_{y\in X} z(y)^{f(y)}$ is continuous. The continuity of G follows from Weierstrass criterion for uniform convergence, since $\sup_{z\in[0,1]^X} \mu_x(f) \prod_{y\in X} z(y)^{f(y)} = \mu_x(f)$ which is summable.

The set $[0,1]^X$ is partially ordered by $z \geq z'$ if and only if $z(x) \geq z'(x)$ for all $x \in X$; by $z > z'$ we mean that $z \geq z'$ and $z \neq z'$. We denote by $\mathbf{0}$ and $\mathbf{1}$ the smallest and largest element of $[0,1]^X$ respectively, that is $\mathbf{0}(x) := 0$ and $\mathbf{1}(x) := 1$ for every $x \in X$. The topological (partially ordered) space $[0,1]^X$ is compact and every monotone sequence has a cluster point, moreover all the intervals $(-\infty, z] \equiv [\mathbf{0}, z]$ and $[z, +\infty) \equiv [z, \mathbf{1}]$ are closed sets whence all the hypotheses of Corollary 2.2 are satisfied. Let us note that $G(\mathbf{1}) = \mathbf{1}$ and G is nondecreasing. From now on we suppose that $\mu_x(\mathbf{0}) > 0$ for some $x \in X$ in order to avoid a trivial case of almost sure survival.

Let $q_n(x)$ be the probability of extinction before or at the n-th generation starting from a single initial particle of type x; and let $q(x)$ be the probability of extinction at any time starting from the same configuration. Note that q_n and q can be viewed as elements of

$[0, 1]^X$. Clearly $q_0 = \mathbf{0}$ and

$$q_{n+1}(x) = \sum_{f \in \Psi} \mu_x(f) \prod_{y \in X} q_n(y)^{f(y)} = G(q_n|x);$$

$$q(x) = \lim_{n \to \infty} q_n(x).$$

According to Proposition 2.1 and Corollary 2.2, q is the smallest fixed point of G and $q < \mathbf{1}$ if and only if

$$G(y) \leq y \quad \text{for some } y < \mathbf{1}. \tag{3.2}$$

Hence, if y satisfies (3.2) then $y(x)$ is an upper bound for $q(x)$. Conversely if we define

$$H(v) := \mathbf{1} - G(\mathbf{1} - v) \tag{3.3}$$

then H is nondecreasing and continuous; moreover if

$$\begin{cases} v_{n+1} = H(v_n), \\ v_0 = \mathbf{1}. \end{cases} \tag{3.4}$$

then $\{v_n\}_{n \in \mathbb{N}}$ is nonincreasing and has a unique cluster point $v := \lim_{n \to \infty} v_n = \mathbf{1} - q$. Clearly $v_n(x)$ can be interpreted as the probability of survival up to the n-th generation for the BRW starting with one particle on x ($v(x)$ being the probability of surviving forever).

Moreover $v > \mathbf{0}$ if and only if $H(y) \geq y$ for some $y \geq \mathbf{0}$. Note that in this case $y(x)$ is a lower bound for $v(x)$. Let G_n and H_n be the n-th iterates of G and H; $H_n(v) = \mathbf{1} - G_n(\mathbf{1} - v)$ and they are continuous and nondecreasing.

Remark 3.1 Let us consider the graph (X, E_μ) where $E_\mu := \{(x, y) \in X^2 : \exists f \in \Psi, f(y) > 0, \mu_x(f) > 0\}$. We call the IBP *irreducible* if and only if the graph (X, E_μ) is connected. It is easy to show that for the extinction probabilities q of an irreducible IBP we have $q < \mathbf{1}$ (that is $v > \mathbf{0}$) if and only if $q(x) < 1$ for all $x \in X$ (that is $v(x) > 0$ for all $x \in X$).

3.1 Infinite-Type Branching Processes Associated to Branching Random Walks

In order to study the weak behavior of the BRW, we associate a discrete-time branching process to the (continuous-time) BRW in such a way that they both survive or both die at the same time. Each particle of the BRW living on a site x will be given the label x which represents its type. We suppose that the BRW starts from a single particle in a vertex x_0; if there are several particles we repeat this construction for each initial particle. The IBP is constructed as follows: the 0th generation is one particle of type x_0; the 1st generation of the IBP is the collection of the children of this particle (ever born): this collection is almost surely finite, say, r_1 particles in the vertex $x_1, \ldots, r_m$ particles in x_m. Thus from the point of view of the IBP the 1st generation is the collection of r_1 particles of type $x_1, \ldots, r_m$ particles of type x_m. Take one particle of type x_1 in the 1-st generation and collect all its children, repeat this for all the particles in the 1st generation: the set of all these new particles is the 2nd generation. Proceeding in the same way we construct the 3rd generation and so on.

Clearly the progeny of the IBP is the same as the progeny of the BRW hence the latter is finite (i.e. the BRW dies out) if and only if the former is finite (i.e. the IBP dies out). The probabilities of extinction of the IBP (that is, the smallest fixed point of the generating

function), regarded as an element of $[0, 1]^X$, coincide with the probabilities of extinction of the BRW.

Let us compute the generating function of this IBP. Roughly speaking, the probability for a particle of type x of having $f(y)$ children of type y for all $y \in X$ (where $f \in \Psi$) is the probability that, for all $y \in X$, a Poisson clock of rate λk_{xy} rings $f(y)$ times before the death of the original particle (i.e. a clock of rate 1). Elementary computations show that

$$\mu_x(f) = \frac{S(f)! \prod_{y \in X} (\lambda k_{xy})^{f(y)}}{(1 + \lambda \sum_{y \in X} k_{xy})^{S(f)+1} \prod_{y \in X} f(y)!}.$$

Recalling (3.1) we have

$$G^\lambda(z|x) = \sum_{f \in \Psi} \frac{S(f)! \prod_{y \in X} (\lambda k_{xy})^{f(y)}}{(1 + \lambda \sum_{y \in X} k_{xy})^{S(f)+1} \prod_{y \in X} f(y)!} \prod_{y \in X} z(y)^{f(y)}$$

$$= \frac{1}{1 + \lambda \sum_{y \in X} k_{xy}} \sum_{i=0}^{+\infty} \sum_{f:S(f)=i} \frac{i!}{\prod_{y \in X} f(y)!} \frac{1}{(1 + \lambda \sum_{y \in X} k_{xy})^i} \prod_{y \in X} (\lambda k_{xy} z(y))^{f(y)}$$

$$= \frac{1}{1 + \lambda \sum_{y \in X} k_{xy}} \sum_{i=0}^{+\infty} \left(\frac{\lambda \sum_{y \in X} k_{xy} z(y)}{1 + \lambda \sum_{y \in X} k_{xy}} \right)^i$$

$$= \frac{1}{1 + \lambda \sum_{y \in X} k_{xy}(1 - z(y))}. \tag{3.5}$$

We note that the quantity λk_{xy} can be interpreted as the expected number of offsprings of type y of a particle of type x. Clearly in this case

$$H^\lambda(v; x) = \frac{\lambda \sum_{y \in X} k_{xy} v(y)}{1 + \lambda \sum_{y \in X} k_{xy} v(y)}.$$

If we define the bounded linear operator $K : l^\infty(X) \mapsto l^\infty(X)$ as $Kv(x) := \sum_{y \in X} k_{xy} v(y)$ then

$$H^\lambda(v) = \frac{\lambda K v}{1 + \lambda K v}, \tag{3.6}$$

hence the functions H_n^λ and G_n^λ from $[0, 1]^X$ into itself are nondecreasing and continuous with respect to $\| \cdot \|_\infty$ for every $n \geq 1$. In particular each iterate H_n^λ can be extended to the positive cone $l_+^\infty(X) := \{v \in l^\infty(X) : v \geq \mathbf{0}\}$. We observe that the operator K preserves $l_+^\infty(X)$.

When there is no ambiguity, we will drop the dependence on λ in these functions. From now on, if not stated otherwise, it will be tacitly understood that G and H are defined by (3.5) and (3.6) respectively.

It is easy to show that K is irreducible (as stated in Sect. 2.1) if and only if the corresponding IBP is irreducible in the sense of Remark 3.1.

4 The Critical Values and the Critical Behaviors

4.1 The Critical Values

In [10] it was proved that, in the irreducible case, $\lambda_s = 1/M_s$ for any graph. In [1] we used a different approach to extend this result to multigraphs; the same arguments hold for weighted graphs (we repeat the proof for completeness). This approach allows us to study the critical behavior when $\lambda = \lambda_s(x)$ (see Theorem 4.7). We observe that in the proof of the following theorem to the BRW we associate a particular branching process which is not the one introduced in Sect. 3.1. The proof relies on the concept of (reproduction) *trail*: see [10] for the definition.

Theorem 4.1 *For every weighted graph* (X, K) *we have that* $\lambda_s(x) = 1/M_s(x)$.

Proof Fix $x \in X$, consider a path $\Pi := \{x = x_0, x_1, \ldots, x_n = x\}$ and define its number of cycles $\mathbb{L}(\Pi) := |\{i = 1, \ldots, n : x_i = x\}|$; the expected number of trails along such a path is $\lambda^n \prod_{i=0}^{n-1} k_{x_i x_{i+1}}$ (i.e. the expected number of particles ever born at x, descending from the original particle at x and whose genealogy is described by the path Π—their mothers were at x_{n-1}, their grandmothers at x_{n-2} and so on). Disregarding the original time scale, to the BRW there corresponds a Galton-Watson branching process: given any particle p in x (corresponding to a trail with n cycles), define its children as all the particles whose trail is a prolongation of the trail of p and is associated with a spatial path with $n + 1$ cycles. Hence a particle is of the k-th generation if and only if the corresponding trail has k cycles; moreover it has one (and only one) parent in the $(k-1)$-th generation. Since each particle behaves independently of the others then the process is Markovian. Thus the BRW survives strongly if and only if this branching process does. The expected number of children of the branching process is the sum over n of the expected number of trails of length n and one cycle, that is $\sum_{n=1}^{\infty} \phi_{x,x}^n \lambda^n = \Phi(x, x|\lambda)$. Thus we have a.s. local extinction if and only if $\Phi(x, x|\lambda) \leq 1$, that is, $\lambda(x) \leq 1/M_s(x)$. □

We turn our attention to the weak critical parameter $\lambda_w(x)$, which, by Corollary 2.2, may be characterized in terms of the function H^λ (defined by (3.6)):

$$\lambda_w(x) = \inf\{\lambda \in \mathbb{R} : \exists v \in l_+^\infty(X), v(x) > 0, H^\lambda(v) \geq v\}$$

$$= \inf\left\{\lambda \in \mathbb{R} : \exists v \in [0, 1]^X, v(x) > 0, \lambda K v \geq \frac{v}{1 - v}\right\}. \tag{4.1}$$

Our goal is to give other characterizations of $\lambda_w(x)$. Theorem 4.2 shows that, for every $n \geq 1$

$$\lambda_w(x) = \inf\{\lambda \in \mathbb{R} : \exists v \in l_+^\infty(X), v(x) > 0, H_n^\lambda(v) \geq v\}$$

$$= \inf\{\lambda \in \mathbb{R} : \exists v \in l_+^\infty(X), v(x) > 0, \lambda^n K^n v \geq v\}; \tag{4.2}$$

thus, by taking $n = 1$ in the previous equation, $\lambda_w(x) = \inf\{\underline{r}_K(v) : v \in l^\infty(X), v(x) = 1\}$ where $\underline{r}_K(v)$ is the *lower Collatz-Wielandt number of* v (see [2, 3, 8]).

We note that (4.2) is particularly useful to compute the value of λ_w (indeed solving the linear inequality therein is easier than solving the nonlinear inequality in (4.1)). Unfortunately the critical (global) survival of the BRW (with one initial particle at x) is equivalent to the existence of a solution of $\lambda_w(x) K v \geq v/(1 - v)$ with $v(x) > 0$ (see Example 3). The existence of a solution of $\lambda_w(x) K v \geq v$ does not imply critical survival.

Theorem 4.2 *Let (X, K) be a weighted graph and let $x \in X$.*

(a) *If $\lambda \leq \lambda_w(x)$ and $v \in [0, 1]^X$ is such that $\lambda K v \geq v/(1 - v)$ then $\inf_{y: x \to y, v(y) > 0} v(y) = 0$.*
(b) *For all $n \in \mathbb{N}$, $n \geq 1$ we have*

$$\lambda_w(x) = \inf\{\lambda \in \mathbb{R} : \exists v \in l_+^\infty(X), v(x) > 0 \text{ such that } H_n^\lambda(v) \geq v\}.$$

(c) *For all $n \in \mathbb{N}$, $n \geq 1$ we have*

$$\lambda_w(x) = \inf\{\lambda \in \mathbb{R} : \exists v \in l_+^\infty(X), v(x) > 0 \text{ such that } \lambda^n K^n v \geq v\}.$$

Proof

(a) Let $X' := \{y \in X : x \to y\}$ and consider $v'(y) := v(y)\mathbb{1}_{X'}(y)$, then $\lambda K v' \geq v'/(1 - v')$ (since $\lambda K v'(y) = \lambda K v(y)$ for all $y \in X'$). Thus we may suppose, without loss of generality, that $X' = X$. For all $t \in [0, 1]$ we have $\lambda K(tv) \geq \frac{tv}{1-tv} \frac{1-tv}{1-v}$ and $v \mapsto \frac{1-tv}{1-v}$ is nondecreasing. By contradiction, suppose that $\inf_{y \in X: v(y) > 0} v(y) = \delta > 0$, hence $\frac{1-tv}{1-v} \geq \frac{1-t\delta}{1-t\delta}\mathbb{1}$ and $(\lambda \frac{1-\delta}{1-t\delta}) K(tv) \geq \frac{tv}{1-tv}$. By (4.1), for all $t \in (0, 1)$, $\lambda > \lambda \frac{1-\delta}{1-t\delta} \geq \lambda_w(x)$.
(b) Define $\lambda_n(x) := \inf\{\lambda \in \mathbb{R} : \exists v \in l_+^\infty(X), v(x) > 0, H_n^\lambda(v) \geq v\}$. Clearly $H^\lambda(v) \geq v$ implies $H_n^\lambda(v) \geq v$, thus $\lambda_w(x) \geq \lambda_n$. If $\lambda > \lambda_n$ then by Corollary 2.2 the sequence $\{\tilde{v}_i\}_{i \in \mathbb{N}}$ defined by $\tilde{v}_0 = \mathbb{1}$, $\tilde{v}_{i+1} = H_n^\lambda(\tilde{v}_i)$ converges monotonically to some $v > \mathbf{0}$, namely $\tilde{v}_i \downarrow v$. But $\tilde{v}_i = v_{ni}$ (for all $i \in \mathbb{N}$) where the nonincreasing sequence $\{v_j\}_{j \in \mathbb{N}}$ is defined by (3.4), whence $v_j \downarrow v$. By (4.1), since $H^\lambda(v) = v$, we get $\lambda \geq \lambda_w(x)$.
(c) Define now $\lambda_n := \inf\{\lambda \in \mathbb{R} : \exists v \in l_+^\infty(X), v(x) > 0 \text{ such that } \lambda^n K^n v \geq v\}$. We prove that $\lambda_n \leq \lambda_w(x)$ for all $n \geq 1$. Indeed, if $\lambda > \lambda_w(x)$, then there exists $\tilde{v}$ such that $\lambda K \tilde{v} \geq \frac{\tilde{v}}{1-\tilde{v}} \geq \tilde{v}$. By induction on n, $\lambda^n K^n \tilde{v} \geq \tilde{v}$, thus, for all n, $\lambda \geq \lambda_n$ which implies $\lambda_n \leq \lambda_w$. On the other hand, if $\lambda > \lambda_n$ then there exists $\lambda' \in [\lambda_n, \lambda)$ such that $(\lambda')^n K^n v \geq v$ for some $v \in l_+^\infty(X)$ such that $v(x) > 0$. If $\varepsilon = \lambda/\lambda' - 1$ and $\delta > 0$ is such that $\|\lambda K H_{n-1}^\lambda(\delta' v)\|_\infty \leq \varepsilon$ for all $\delta' \in (0, \delta]$ (which is possible since H_n^λ is continuous and $H_n^\lambda(\mathbf{0}) = \mathbf{0}$) then we have that $H_n^\lambda(\delta' v) \geq (\lambda/(1 + \varepsilon)) K H_{n-1}^\lambda(\delta' v)$. By induction on n and since K is a positive operator there exists $\tilde{\delta} > 0$ such that $H_n^\lambda(\tilde{\delta} v) \geq (\lambda/(1 + \varepsilon))^n K^n H_0^\lambda(\tilde{\delta} v) = (\lambda')^n K^n(\tilde{\delta} v) \geq \tilde{\delta} v$ whence $\lambda \geq \lambda_w(x)$ by (b) and this implies $\lambda_n \geq \lambda_w(x)$. $\qquad\square$

The following theorem improves Lemma 3.2 of [1].

Theorem 4.3 *For every weighted graph (X, K) we have that $\lambda_w(x) \geq 1/M_w^-(x)$.*

Proof Let $\lambda < 1/M_w^-(x)$. If there exists $v \in l_+^\infty(X)$ such that $\lambda K v \geq \frac{v}{1-v} \geq v$, then for all $n \in \mathbb{N}$ we have $\lambda^n K^n v \geq v$. Thus $\|v\|_\infty \lambda^n \sum_{y \in X} k_{xy}^n \geq \lambda^n K^n v(x) \geq v(x)$, but, since $\lambda \liminf_n \sqrt[n]{\sum_{y \in X} k_{xy}^n} < 1$, we have that $\liminf_n \lambda^n \sum_{y \in X} k_{xy}^n = 0$, whence $v(x) = 0$. By (4.1), $\lambda \leq \lambda_w(x)$. $\qquad\square$

Remark 4.4 Let us focus on the particular case where X is finite. Clearly if K is irreducible, then $\lambda_w = \lambda_s = 1/M_w = 1/M_s = 1/M_w^-$ and these parameters do not depend on the site of the initial particle.

If X is finite, but K is not irreducible, it may happen that $\lambda_w(x) \neq \lambda_s(x)$ and also $\lambda_w(x) \neq \lambda_w(y)$ (although $\lambda_w(x) \leq \lambda_w(y)$ for all y such that $x \to y$).

Moreover in the finite case $\lambda_w(x, [x]) = 1/M_w^-(x, [x])$ (where by adding $[x]$ we consider the parameters corresponding to the process restricted to $[x]$, namely $([x], K|_{[x] \times [x]})$).

From this and the fact that the BRW starting from one particle in x survives globally if and only if it survives (locally and globally) in at least one irreducible class, it follows that $\lambda(x, X) = \min\{1/M_w^-(y, [y]) : x \to y\}$. By induction on the number of equivalence classes, it is not difficult to prove that $M_w^-(x, X) = \max\{M_w^-(y, [y]) : x \to y\}$ which proves, for finite weighted graphs, that $\lambda_w(x, X) = 1/M_w^-(x)$.

As for the critical behavior, the $\lambda_w(x)$-BRW dies out (globally, thus locally) almost surely. Indeed if $\lambda_w(x) < \lambda_w(y, [y]) = \lambda_s(y)$ it cannot survive confined to $[y]$. If $\lambda_w(x) = \lambda_w(y, [y])$ then according to *(a)* of Theorem 4.2 the probabilities of survival v for the process confined to $[y]$ satisfy $\inf_{z \in [y]} v(z) = 0$. Being $[y]$ finite and irreducible, this means that $v(z) = 0$ for all $z \in [y]$.

We say that (X, K) is *locally isomorphic* to $(Y, \widetilde{K})$ if and only if there exists a surjective map $f : X \to Y$ such that $\sum_{z \in f^{-1}(y)} k_{xz} = \widetilde{k}_{f(x)y}$ for all $x \in X$ and $y \in Y$. An (X, K) which is locally isomorphic to some $(Y, \widetilde{K})$ "inherits" its M_w^-s and λ_ws (in a sense which is clear in the proof of the following proposition).

Proposition 4.5 *If Y is a finite set and (X, K) is locally isomorphic to $(Y, \widetilde{K})$ then $\lambda_w(x) = 1/M_w^-(x)$.*

Proof The definition of the map f immediately implies that $\sum_{z \in X} k_{xz}^n = \sum_{y \in Y} \widetilde{k}_{f(x)y}^n$, hence $M_w^-(x, X) = M_w^-(f(x), Y)$. Moreover $\lambda_w(x, X) = \lambda_w(f(x), Y)$. Indeed it is easy to prove that $\lambda_w(x, X) \ge \lambda_w(f(x), Y)$. Conversely, if $\lambda > \lambda_w(f(x), Y)$ and $\widetilde{v}$ is such that $\lambda \widetilde{K}\widetilde{v} \ge \widetilde{v}$ then we define $v(x) := \widetilde{v}(f(x))$. Clearly $\widetilde{K}\widetilde{v}(f(x)) = Kv(x)$ hence $\lambda \ge \lambda_w(x, X)$. Remark 4.4 yields the conclusion. □

Examples of BRWs (X, K) which are locally isomorphic to some finite $(Y, \widetilde{K})$ are BRWs where $\sum_{z \in X} k_{xz}$ does not depend on x (in this case $Y = \{y\}$ is a singleton and $\widetilde{k}_{yy} = \sum_{z \in X} k_{xz}$). Another example is given by *quasi-transitive* BRWs, that is, there exists a finite $X_0 \subset X$ such that for any $x \in X$ there is a bijective map $\gamma_x : X \to X$ satisfying $\gamma_x^{-1}(x) \in X_0$ and $k_{yz} = k_{\gamma_x y \gamma_x z}$ for all y, z (in this case $Y = X_0$ and $\widetilde{k}_{wz} = \sum_{y:y=\gamma_y(z)} k_{wy}$).

Let us consider now the irreducible case; since $\lambda_w(x)$ and $M_w^-(x)$ do not depend on x let us write λ_w and M_w^- instead. Note that the characterization of λ_w can be written as

$$\lambda_w = \inf\{\lambda \in \mathbb{R} : \exists v \in l^\infty(X), v > \mathbf{0}, H^\lambda(v) \ge v\}$$
$$= \inf\{\lambda \in \mathbb{R} : \exists v \in l^\infty(X), v > \mathbf{0}, \lambda^n K^n v \ge v\}$$
$$= \inf\{\lambda \in \mathbb{R} : \exists v \in l^\infty(X), v > \mathbf{0}, H_n^\lambda(v) \ge v\}, \tag{4.3}$$

where the requirement $v > \mathbf{0}$ seems less restrictive than $v(x) > 0$ for all $x \in X$, which is the one we would expect in view of (4.1). Nevertheless by Remark 3.1 it follows that if there exists $v > \mathbf{0}$ satisfying one of the inequalities in (4.1), then there exists a solution v' of the same inequality with $v'(x) > 0$ for all $x \in X$.

Proposition 4.6 *Let (X, K) be an irreducible weighted graph. If for all $\varepsilon > 0$ there exists N such that $\sum_{y \in X} k_{xy}^N \ge (M_w^- - \varepsilon)^N$, for all $x \in X$, then $\lambda_w = 1/M_w^-$.*

Proof Let $\lambda > 1/M_w^-$. Choose ε such that $\lambda(M_w^- - \varepsilon) > 1$. Then $\lambda^N K^N \mathbf{1}(x) = \lambda^N \sum_{y \in X} k_{xy}^N \ge (\lambda(M_w^- - \varepsilon))^N > 1$. Hence by Theorem 4.2 $\lambda > \lambda_w$. Theorem 4.3 yields the conclusion. □

We note that if (X, K) is irreducible and satisfies the hypothesis of Proposition 4.5, then Proposition 4.6 provides an alternative proof of $\lambda_w = 1/M_w^-$. For an example (which is not locally isomorphic to a finite weighted graph), where one can use Proposition 4.6, see Example 3 in [1] (which is a BRW on a particular radial tree).

4.2 The Critical Behavior

Theorem 4.7 *For each weighted graph (X, K) if $\lambda = \lambda_s(x)$ then the λ-BRW starting from one particle at $x \in X$ dies out locally almost surely.*

Proof Recall that (see the proof of Theorem 4.1) the $\lambda(x)$-BRW survives locally if and only if the Galton-Watson branching process with expected number of children $\Phi(x, x|\lambda)$ does. Since $\Phi(x, x|1/M_s) \le 1$ and $\lambda_s(x) = 1/M_s$ then there is a.s. local extinction at $\lambda_s(x)$. □

Theorem 4.8 *If Y is a finite set and (X, K) is locally isomorphic to $(Y, \widetilde{K})$ then the $\lambda_w(x)$-BRW starting from one particle at $x \in X$ dies out globally almost surely.*

Proof It is clear that the $\lambda_w(x)$-BRW on X dies out if and only if the $\lambda_w(x)$-BRW on Y does. Remark 4.4 yields the conclusion. □

5 Examples

We start by giving an example of an irreducible IBP where, although the expected number of children of each particle is less than 1, nevertheless the colony survives with positive probability.

Example 1 Let $X = \mathbb{N}$, $\{p_n\}_{n \in \mathbb{N}}$ be a sequence in $[0, 1)$ and suppose that a particle of type $n \ge 1$ at the end of its life has one child of type $n + 1$ with probability $1 - p_n$, one child of type $n - 1$ with probability $p_n/2$ (if $n = 0$ then it has one child of type 0 with probability $p_0/2$) and no children with probability $p_n/2$. The generating function G can be explicitly computed

$$G(z|n) = \begin{cases} \frac{p_n}{2} + \frac{p_n}{2} z(n-1) + (1 - p_n)z(n+1), & n \ge 1, \\ \frac{p_0}{2} + \frac{p_0}{2} z(0) + (1 - p_0)z(1), & n = 0. \end{cases}$$

This process clearly dominates the (reducible) one where a particle of type n at the end of its life has one child of type $n + 1$ with probability $1 - p_n$ and no children with probability p_n. The latter process has generating function $\widetilde{G}(z|n) = p_n + (1 - p_n)z(n + 1)$. By induction it is easy to show that the probabilities of extinction before or at generation n of the second process are $q_n(j) = 1 - \prod_{i=j}^{j+n-1}(1 - p_i)$ for all $n \ge 1$; hence it survives with positive probability, that is $q_n(0) \not\to 1$ as $n \to \infty$, if and only if $\sum_{i=1}^{\infty} p_i < +\infty$.

In all the following examples, $X = \mathbb{N}$ and $k_{ij} = 0$ whenever $|i - j| > 1$. Although this looks quite restrictive, one quickly realizes that many BRWs are locally isomorphic to BRWs of this kind. For instance, every BRW on a homogeneous tree of degree m (with $k_{ij} = 1$ on each edge) is locally isomorphic to the BRW on $\mathbb{N}$ with $k_{01} := m$, $k_{n\,n+1} := m - 1$, $k_{n\,n-1} := 1$ and 0 otherwise. More generally any *radial BRW on a radial tree* is locally isomorphic to a BRW on $\mathbb{N}$. Indeed a general radial BRW on a radial tree is constructed as

follows: let us consider two positive real sequences $\{k_n^+\}_{n\in\mathbb{N}}$, $\{k_n^-\}_{n\in\mathbb{N}}$ and a positive integer valued sequence $\{a_n\}_{n\in\mathbb{N}}$. By construction, the root of the tree is some vertex o which has a_0 neighbors and the rates are $k_{ox} := k_0^+$, $k_{xo} := k_0^-$ for all neighbors x. Each vertex x at distance 1 from o has $1 + a_1$ neighbors (one is o) and we set $k_{xy} := k_1^+$ and $k_{yx} := k_1^-$ for all its a_1 neighbors y at distance 2 from o. Now each vertex at distance 2 from o has $1 + a_2$ neighbors, an outward rate k_2^+ and an inward rate k_2^- and so on. This BRW is clearly locally isomorphic to (therefore it has the same global behavior of) the BRW on $\mathbb{N}$ with $k_{n\,n+1} := a_n k_n^+$, $k_{n+1\,n} := k_n^-$ and 0 otherwise.

The next one is an example of a BRW on $\mathbb{N}$ which is not irreducible and where $\lambda_w > 1/M_w$. This answers an open question raised in [1].

Example 2 Let $\{k_n\}_{n\in\mathbb{N}}$ be a bounded sequence of positive real numbers and let us consider the BRW on $\mathbb{N}$ with rates $k_{ij} := k_i$ if $i = j - 1$ and 0 otherwise. By using (3.4) and (3.6) one can show that $v_n(i) = \lambda^n \beta_{i+n}/(1 + \sum_{r=1}^n \lambda^r \beta_{i+n}/\beta_{i+n-r})$ where $\beta_n := \prod_{i=0}^{n-1} k_i$.

In order to prove that $\lambda_w(i) = 1/\liminf_n \sqrt[n]{\beta_{n+i}/\beta_i} = 1/M_w^-(i)$ (which does not depend on i, though the BRW is not irreducible) one may either study the behavior of $\{v_n\}_{n\in\mathbb{N}}$ above or, which is simpler, use Theorem 4.2. Indeed, without loss of generality, we just need to prove that for all $\lambda > 1/\liminf_n \sqrt[n]{\beta_n}$ it is possible to solve the inequality $\lambda K v \geq v$ for some $v \in l^\infty(X)$, $v > \mathbf{0}$. One can easily check that $v(n) := 1/(\lambda^n \beta_n)$ is a solution; since $\lambda > 1/\liminf_n \sqrt[n]{\beta_n}$ we have that $\lim_n v(n) = 0$ and then $v \in l^\infty(X)$.

Note that $\lambda_w = 1/M_w^-$ which may be different from $1/M_w$ with the following choice of the rates. It suffices to define big intervals of consecutive vertices where $k_{i\,i+1} = 1$, followed by bigger intervals of vertices where $k_{i\,i+1} = 2$ and so on. The result is a BRW where $M_w = \limsup_n \sqrt[n]{\beta_n} = 2$ while $M_w^- = \liminf_n \sqrt[n]{\beta_n} = 1$.

Although this BRW is not irreducible, it is clear that a slight modification (that is, adding a small backward rate as in the following example) does not modify significantly the behavior of the process and allows to construct an irreducible example with the same property. Finally, the last example shows that the weak critical survival is possible (while, according to Theorem 4.7, any strong critical BRW dies out locally).

Example 3 Let $X := \mathbb{N}$ and K be defined by $k_{01} := 2$, $k_{n\,n+1} := (1 + 1/n)^2$, $k_{n+1\,n} := 1/3^{n+1}$ and 0 otherwise. Hence the inequality $\lambda K v \geq v/(1 - v)$ becomes

$$\begin{cases} 2\lambda v(1) \geq v(0)/(1 - v(0)), \\ \lambda(v(n+1)(1 + 1/n)^2 + v(n-1)/3^n) \geq v(n)/(1 - v(n)). \end{cases}$$

Clearly $v(0) = 1/2$ and $v(n) := 1/(n + 1)$ (for all $n \geq 1$) is a solution for all $\lambda \geq 1$. If $\lambda < 1$ then one can prove by induction that a solution must satisfy $v(n+1)/v(n) \geq \frac{1}{\lambda}(\frac{n}{n+1})^2(1 - \frac{1}{2^n})$ for all $n \geq 2$. Thus $v(n+1)/v(n)$ is eventually larger than $1 + \varepsilon$ for some $\varepsilon > 0$, hence either $v = \mathbf{0}$ or $\lim_n v(n) = +\infty$. This implies that $\lambda_w = 1$ and there is global survival if $\lambda = \lambda_w$.

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